How complex is the topological conjugacy relation of transitive maps?

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Goals

How complex is the **homeomorphism** relation

on various subclasses of compact metric spaces?

resp.

How complex is the **conjugacy** relation

of continuous selfmaps on such spaces?

Framework: invariant descriptive set theory

Definition

- Let X, Y be sets,
- let E, F be equivalence relations (ER) on X and Y respectively.

A map $\varphi\colon X \to Y$ is called a reduction from E to F if for every two points $x, x' \in X$ we have $xEx' \iff \varphi(x)F\varphi(x')$.

Definition

- Let X, Y be Polish spaces (standard Borel spaces),
- let E, F be ERs on X, Y respectively.

We say that E is Borel reducible to F, ($E \leq_B F$), if there is a Borel measurable reduction from E to F.

We say that E is Borel bireducible with F, ($E \sim_B F$), if $E \leq_B F$ and $F \leq_B E$.

Complexity degrees and benchmarks

- universal analytic ER: E_∞
  - the most complex among all analytic ERs
  - isomorphism ER of separable Banach spaces (Ferenczi, Louveau, Rosendal)
- universal orbit ER: E_{G_∞}
  - the most complex among all orbit ERs induced by Polish groups
  - isometry ER of complete separable metric spaces (Gao, Kechris)
  - isometry ER of separable Banach spaces (Melleray)
- S_∞-universal orbit ER: E_{S_∞}
  - the most complex among all orbit ERs generated by the group S_∞
  - isomorphism ER of countable graphs, or countable linear orders
  - isomorphism ER of most countable structures (Friedman, Stanley)
- equality of countable sets: $E_{=^+}$
  - the most complex Π^0_3 ER induced by S_∞
  - $(x_n), (y_n) \in \mathbb{R}^\mathbb{N}$ are equivalent iff $\{x_n : n \in \mathbb{N}\} = \{y_n : n \in \mathbb{N}\}$

$$E_{=^+} \lneq_B E_{S_\infty} \lneq_B E_{G_\infty} \lneq_B E_\infty$$

Setting and restrictions

- compact metric spaces
- the hyperspace $K(X)$ of compact subsets of X, Vietoris topology
- Hilbert cube Q contains a copy of every compact metric space
- the homeo ER:

$$\{(K, L) \in K(Q)^2 : K \text{ homeo to } L\}$$

- $C(X) = \{f : X \to X, \text{ continuous}\}$ with the uniform topology
- $H(X) = \{f \in C(X) : f \text{ invertible}\}$, a Polish subspace of $C(X)$
- the conjugacy ER:

$$\{(f, g) \in C(X)^2 : \exists h \in H(X) : h \circ f = g \circ h\}$$

Infinite-dimensional topological classes

Theorem

2016 Zielinski: Homeo ER on compacta $\sim_B E_{G_\infty}$
based on Sabok: isomorphism ER of C^**-algebras*
resp. affine homeo of Choquet simplices

2017 Chang, Gao: Homeo ER on continua $\sim_B E_{G_\infty}$

2018 Cieśla: Homeo ER on LC continua $\sim_B E_{G_\infty}$

2018 Krupski, V.: Homeo ER on AR $\sim_B E_{G_\infty}$

locally connected (LC) = having a base formed by connected sets
absolute retract (AR) = retract of the Hilbert cube $[0,1]^{\mathbb{N}}$

It is not clear to me, whether we can get E_{G_∞} on some subclass of countably dimensional compacta (coanalytic set).

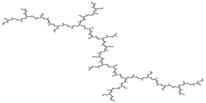
Low-dimensional topological classes

Definition

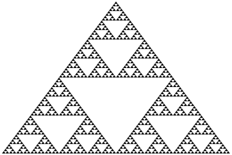
- 0-dimensional compacta
  $=$ closed subspaces of the Cantor set
- dendrites
  $=$ 1-dimensional AR
- rim-finite (RF) continua
  $=$ have a base with finite boundaries
- rim-finite compacta
- $\mathrm{AR}(\mathbb{R}^2)$ $=$ AR in the plane

Dendrite $\iff$ RF and $\mathrm{AR}(\mathbb{R}^2)$

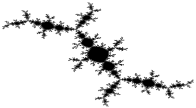
0-dimensional $\implies$ RF



a dendrite



a rim-finite continuum



$\mathrm{AR}(\mathbb{R}^2)$

Low dimensional case

We are omitting: 'Homeomorphism ER on'

Theorem

2001 Camerlo, Gao: 0-dimensional compacta $\sim_B E_{S_\infty}$ *(Stone duality)*

2005 Camerlo, Darji, Marcone: dendrites $\sim_B E_{S_\infty}$

2018 Krupski, V.: rim-finite continua $\sim_B E_{S_\infty}$

2019 Dudák, V.: $\mathrm{AR}(\mathbb{R}^2) \sim_B E_{S_\infty}$ *(reduction to the boundary)*

2019 Dudák, V.: $\mathrm{AR}(\mathbb{R}^3)$, $\mathrm{LC}(\mathbb{R}^2) \not\sim_B E_{S_\infty}$

2018 Krupski, V.: rim-finite compacta $\not\sim_B E_{S_\infty}$ *(Hjorth: turbulence)*

dendrites = 1-dimensional AR
rim-finite = having a base with finite boundaries

Some natural class with complexity strictly between E_{S_∞} and E_{G_∞}?

Conjugacy ER

2000 Hjorth: conjugacy ER of interval homeos $\sim_B E_{S_\infty}$

2001 Camerlo, Gao: conjugacy of Cantor set homeos $\sim_B E_{S_\infty}$

2023 Bruin, V.: conjugacy of interval maps $\sim_B E_{S_\infty}$

conjugacy of Hilbert cube homeos: $\sim_B E_{G_\infty}$ (fixed points)

Hjorth's conjecture

Every ER which is induced by a continuous action of the group $\mathcal{H}([0,1])$ is Borel reducible to E_{S_∞}, i.e.

$$E_{\mathcal{H}([0,1])} \leq_B E_{S_\infty}.$$

Work in progress with M. Hevessy (towards confirming the conjecture)

Conjugacy of transitive homeos

M. Foreman (2022): How complex is the conjugacy ER of transitive Cantor set homeos?
L. Ding (Nankai Logic seminar): How complex is the conjugacy relation of transitive homeos on compact metric spaces?

- We answer both questions by essentially the same technique, in spite of that the complexity levels differ.
- Moreover, for the second question it is enough to consider Hilbert cube homeos only.

Transitive homeos of the Cantor set

Theorem

Let F be the conjugacy ER of transitive homeos of the Cantor set. Then $F \sim_B E_{S_\infty}$.

Proof.

- $F \leq_B$ conjugacy ER of all homeos $\leq_B E_{S_\infty}$. Thus $F \leq_B E_{S_\infty}$.
- Consider zero-dimensional compacta with at least two points.
- Let E be the homeo ER of such spaces (hyperspace coding).
- Camerlo, Gao: $E_{S_\infty} \leq_B E$. It is enough to prove $E \leq_B F$.
- $\varphi : A \mapsto (A^{\mathbb{Z}}, \sigma_A)$
- A and B are homeomorphic iff $\varphi(A)$ and $\varphi(B)$ are conjugate.
- Note that $A^{\mathbb{Z}}$ is homeo to the Cantor set.
- Cheating: not a fixed Cantor set in the range. Burgess selection theorem does the job. Consequently $E_{S_\infty} \leq_B F$.

Transitive homeos of the Hilbert cube

Theorem

Let F be the conjugacy ER of transitive homoes of the Hilbert cube. Then $F \sim_B E_{G_\infty}$.

Proof.

- F is induced by action of the Polish group $H(Q)$, thus $F \leq_B E_{G_\infty}$.
- Consider nondegenerate retracts of the Hilbert cube (AR).
- Let E be the homeo ER of such spaces.
- Krupski, V.: $E_{G_\infty} \leq_B E$. It is enough to prove $E \leq_B F$.
- $\varphi : A \mapsto (A^{\mathbb{Z}}, \sigma_A)$
- A and B are homeomorphic iff $\varphi(A)$ and $\varphi(B)$ are conjugate.
- Note that $A^{\mathbb{Z}}$ is homeo to the Hilbert cube (Toruńczyk).
- Cheating: not a fixed Hilbert cube in the range. Burgess selection theorem does the job. Consequently $E_{G_\infty} \leq_B F$.

Conjugacy of transitive maps on $[0, 1]$ or circle

Theorem

Conjugacy of transitive maps on $[0, 1]$ *or circle* $\leq_B E_{=^+}$.

Proof.

- Let $f : [0, 1] \to [0, 1]$ be transitive.
- $P_n = \{x \in [0, 1] : f^n(x) = x\}$... closed and nowhere dense.
- Let $P_f = \bigcup Acc(P_n)$... countable
- Periodic points of f are dense, hence P_f is dense.
- Let $\varphi(f) = \{((f^k(x_i)?f^l(x_j))_{k,l,i,j} : x_1, \ldots x_n \in P_f\}$ where ? is one of $<, =, >$.
- φ can be coded to obtain a Borel reduction to $E_{=^+}$.

□

Can we get $\sim_B$ in the theorem? Modifying Denjoy circle homeo?

Conjugacy of minimal homeos - open questions

1. How complex is the conjugacy relation of minimal Cantor set homeos?

- Bratteli diagrams…exceptional point
- Kaya 2017: pointed minimal Cantor set homeos
- Deka, García-Ramos, Kasprzak, Kunde, Kwietniak 202?: non Borel

2. How complex is the conjugacy relation of minimal homeos?

- Li, Peng 2024: Conjugacy ER of minimal homeos is not $\leq_B E_{S_\infty}$.
- Sabok conjecture: $\sim_B E_{G_\infty}$

3. Is there a concrete compact space to which Question 2 could be restricted? Perhaps the infinite dimensional torus?

THANK YOU FOR YOUR ATTENTION